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Several generalizations and applications of incomplete poly-Bernoulli polynomials and incomplete poly-Cauchy polynomials

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1 Introduction

The poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ ([6]) are defined by the generating function

$$\frac{\mathrm{Li}_\mu\bigl(1-e^{-t}\bigr)}{1-e^{-t}}e^{-tx}=\sum_{n=0}^\infty B_n^{(\mu)}(x)\frac{t^n}{n!}\,, \tag{1}$$

where

$$\mathrm{Li}_\mu(z)=\sum_{m=1}^\infty\frac{z^m}{m^\mu}$$

is the μ-th polylogarithm function. Note that e^{-tx} on the left-hand side is replaced by e^{tx} in [1, 2]. When $x=0$, $B_n^{(\mu)}(0)=B_n^{(\mu)}$ are the poly-Bernoulli numbers ([11]). When $x=0$ and $\mu=1$, $B_n^{(1)}(0)=B_n$ are the classical Bernoulli numbers, defined by the generating function

$$\frac{t}{1-e^{-t}}=\sum_{n=0}^\infty B_n\frac{t^n}{n!}\,.$$

The generating function of poly-Cauchy polynomials $c_n^{(\mu)}(x)$ ([12, Theorem 2]) is given by

$$\frac{\mathrm{Lif}_\mu\bigl(\log(1+t)\bigr)}{(1+t)^x}=\sum_{n=0}^\infty c_n^{(\mu)}(x)\frac{t^n}{n!}\,, \tag{2}$$

where

$$\mathrm{Lif}_\mu(z)=\sum_{n=0}^\infty\frac{z^n}{n!(n+1)^\mu}$$

is the *polylogarithm factorial* or *polyfactorial* function. Note that $(1+t)^x$ on the left-hand side is replaced by $(1+t)^{-x}$ in [10]. When $x=0$, $c_n^{(\mu)}(0)=c_n^{(\mu)}$ are the poly-Cauchy numbers $c_n^{(\mu)}$ ([12, Theorem 2]), given by

$$\mathrm{Lif}_\mu\bigl(\log(1+t)\bigr)=\sum_{n=0}^\infty c_n^{(\mu)}\frac{t^n}{n!}\,. \tag{3}$$

When $x=0$ and $\mu=1$, $c_n^{(1)}(0)=c_n$ are the classical Cauchy numbers, defined by the generating function

$$\frac{t}{\log(1+t)}=\sum_{n=0}^\infty c_n\frac{t^n}{n!}\,. \tag{4}$$

In this paper, by using the restricted and associated Stirling numbers of the first kind, we define the restricted and associated poly-Cauchy polynomials. By using the restricted and associated Stirling numbers of the second kind, we define the restricted and associated poly-Bernoulli polynomials. These polynomials are generalizations of original poly-Cauchy polynomials and original poly-Bernoulli polynomials, respectively. We also study their characteristic and combinatorial properties.

2 Restricted and associated Stirling numbers of the second kind

In place of the classical Stirling numbers $\left\{ n \atop k \right\}$ we substitute the restricted Stirling numbers and the associated Stirling numbers, denoted by

$$\left\{ n \atop k \right\}_{\le m} \quad \text{and} \quad \left\{ n \atop k \right\}_{\ge m}\,.$$

Some combinatorial and modular properties of these numbers can be found in [21], and other properties can be found in the papers from the list of references of [21]. The generating functions of these numbers are given by

$$\sum_{n=k}^{mk}\left\{ n \atop k \right\}_{\le m}\frac{x^n}{n!}=\frac{1}{k!}(E_m(x)-1)^k \tag{5}$$

and

$$\sum_{n=mk}^{\infty} \left\{ {n \atop k} \right\}_{\geq m} \frac{x^n}{n!} = \frac{1}{k!}\left(e^x - E_{m-1}(x)\right)^k \tag{6}$$

respectively, where

$$E_m(t) = \sum_{k=0}^{m} \frac{t^k}{k!}$$

is the mth partial sum of the exponential function. These give the number of the k-partitions of an n-element set, such that each block contains at most or at least m elements, respectively. Notice that as particular cases, these numbers when $m = 2$ have been considered by several authors (e.g., [5, 8, 22, 23]). Since the generating function of $\left\{ {n \atop k} \right\}$ is given by

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!} \tag{7}$$

(see e.g., [9]), by $E_\infty(x) = e^x$ and $E_0(x) = 1$, we have

$$\left\{ {n \atop k} \right\}_{\leq \infty} = \left\{ {n \atop k} \right\}_{\geq 1} = \left\{ {n \atop k} \right\}.$$

3 Restricted and associated Stirling numbers of the first kind

Denote by $\left[{n \atop k} \right]$ the (unsigned) Stirling numbers of the first kind, arising as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{i=0}^{n} \left[{n \atop i} \right] x^i.$$

the generating function of $\left[{n \atop k} \right]$ is given by

$$\sum_{n=k}^{\infty} \left[{n \atop k} \right] \frac{x^n}{n!} = \frac{\left(-\log(1-x)\right)^k}{k!}.$$

In place of the classical (unsigned) Stirling numbers of the first kind $\left[{n \atop k}\right]$, we substitute the (unsigned) restricted Stirling numbers of the first kind and the (unsigned) associated Stirling numbers of the first kind, denoted by

$$\left[{n \atop k}\right]_{\le m} \quad \text{and} \quad \left[{n \atop k}\right]_{\ge m},$$

respectively. The associated Stirling number of the first kind $\left[{n \atop k}\right]_{\ge m}$ equals the number of permutations of a set N ($|N| = n$) with k orbits such that each block contains at least m elements ([5, p.256–257], [23]). The restricted Stirling number of the first kind $\left[{n \atop k}\right]_{\le m}$ equals the number of permutations of n with k orbits such that each block contains at most m elements.

For $k \ge 1$ and $m \ge 1$, the generating functions of the restricted Stirling numbers of the first kind $\left[{n \atop k}\right]_{\le m}$ and the associated Stirling numbers of the first kind $\left[{n \atop k}\right]_{\ge m}$ are given by the following ([4, p. 467, (12.8)]). Denote the mth partial sum of the logarithm function by

$$F_m(t) = \sum_{k=1}^{m} (-1)^{k+1} \frac{t^k}{k}. \tag{8}$$

$$\sum_{n=0}^{\infty} \left[{n \atop k}\right]_{\le m} \frac{x^n}{n!} = \frac{1}{k!}\bigl(-F_m(-x)\bigr)^k, \tag{9}$$

$$\sum_{n=0}^{\infty} \left[{n \atop k}\right]_{\ge m} \frac{x^n}{n!} = \frac{1}{k!}\bigl(-\log(1-x) + F_{m-1}(-x)\bigr)^k. \tag{10}$$

It is trivial to see that

$$\left[{n \atop k}\right]_{\le \infty} = \left[{n \atop k}\right]_{\ge 1} = \left[{n \atop k}\right].$$

4 Incomplete poly-Cauchy polynomials

In [10], the concept of the poly-Cauchy polynomials is introduced by replacing x by $-x$ in the definition. For integers n and μ with $n \ge 0$ and $\mu \ge 1$, define the *poly-Cauchy polynomials of the first kind* $c_n^{(\mu)}(x)$ ($\mu \ge 1$) as

$$c_n^{(\mu)}(x) = \underbrace{\int_0^1 \cdots \int_0^1}_{\mu} (t_1 t_2 \cdots t_\mu - x)(t_1 t_2 \cdots t_\mu - x - 1) \cdots (t_1 t_2 \cdots t_\mu - x - n + 1) dt_1 dt_2 \cdots dt_\mu .$$

Then, $c_n^{(\mu)}(x)$ can be expressed in terms of the Stirling numbers of the first kind $\left[{n \atop k} \right]$:

$$c_n^{(\mu)}(x) = \sum_{k=0}^{n} \left[{n \atop k} \right] (-1)^{n-k} \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}$$

([10, Theorem 1]).

Hence, it is natural to define two types of *incomplete poly-Cauchy polynomials of the first kind* $c_{n,\le m}^{(\mu)}(x)$ and $c_{n,\ge m}^{(\mu)}(x)$ by

$$c_{n,\le m}^{(\mu)}(x) = \sum_{k=0}^{n} \left[{n \atop k} \right]_{\le m} (-1)^{n-k} \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \tag{11}$$

$$c_{n,\ge m}^{(\mu)}(x) = \sum_{k=0}^{n} \left[{n \atop k} \right]_{\ge m} (-1)^{n-k} \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \tag{12}$$

respectively.

The generating function of poly-Cauchy polynomials is given by (3). The generating functions of two types of incomplete poly-Cauchy polynomials of the first kind can be also given by using the polylogarithm factorial functions in terms of the mth partial sum of the logarithm function $F_m(t)$.

Theorem 1. *For integers $n \ge 1$, $m \ge 1$ and $\mu \ge 1$ we have*

$$\frac{\mathrm{Lif}_\mu\big(F_m(t)\big)}{e^{xF_m(t)}} = \sum_{n=0}^{\infty} c_{n,\le m}^{(\mu)}(x) \frac{t^n}{n!}, \tag{13}$$

$$\frac{\mathrm{Lif}_\mu\big(\log(1+t) - F_{m-1}(t)\big)}{e^{x(\log(1+t) - F_{m-1}(t))}} = \sum_{n=0}^{\infty} c_{n,\ge m}^{(\mu)}(x) \frac{t^n}{n!}. \tag{14}$$

Remark. Since $F_\infty(t) = \log(1+t)$ and $F_0(t) = 0$, if we take $m \to \infty$ in (13) and $m = 1$ in (14), both of (13) and (14) are reduced to the generating function in ([17, Theorem 3]).

The generating function of two types of the incomplete poly-Cauchy polynomials of the first kind can be written in the form of iterated integrals.

Corollary 1. *For $\mu \geq 1$ we have*

$$\frac{1}{e^{xF_m(t)}F_m(t)} \underbrace{\int_0^t \frac{1-(-t)^m}{(1+t)F_m(t)} \cdots \int_0^t \frac{1-(-t)^m}{(1+t)F_m(t)}}_{\mu-1} (e^{F_m(t)} - 1) \underbrace{dt \cdots dt}_{\mu-1}$$
$$= \sum_{n=0}^{\infty} c_{n,\leq m}^{(\mu)}(x) \frac{t^n}{n!}, \tag{15}$$

$$\frac{1}{e^{x(\log(1+t)-F_{m-1}(t))}\big(\log(1+t) - F_{m-1}(t)\big)}$$
$$\times \underbrace{\int_0^t \frac{(-t)^{m-1}}{(1+t)\big(\log(1+t) - F_{m-1}(t)\big)} \cdots \int_0^t \frac{(-t)^{m-1}}{(1+t)\big(\log(1+t) - F_{m-1}(t)\big)}}_{\mu-1}$$
$$\times \left(e^{\left(\log(1+t) - F_{m-1}(t)\right)} - 1\right) \underbrace{dt \dots dt}_{\mu-1} = \sum_{n=0}^{\infty} c_{n,\geq m}^{(\mu)}(x) \frac{t^n}{n!}. \tag{16}$$

Remark. If we take $m \to \infty$ in (15) and $m = 1$ in (16), both of (15) and (16) are reduced to the generating function in [10, Corollary 1]. When $x = 0$, Corollary (1) is reduced to [15, Corollary 1].

For integers n and μ with $n \geq 0$ and $\mu \geq 1$, define the *poly-Cauchy polynomials of the second kind* $\widehat{c}_n^{(\mu)}(x)$ $(\mu \geq 1)$ as

$$\widehat{c}_n^{(\mu)}(x) = \underbrace{\int_0^1 \dots \int_0^1}_{\mu} (-t_1 t_2 \cdots t_\mu + x)(-t_1 t_2 \cdots t_\mu + x - 1)$$
$$\cdots (-t_1 t_2 \cdots t_\mu + x - n + 1) dt_1 dt_2 \cdots dt_\mu .$$

Then, $\widehat{c}_n^{(\mu)}(x)$ can be expressed in terms of the Stirling numbers of the first kind $\left[{n \atop k}\right]$:

$$\widehat{c}_n^{(\mu)}(x) = \sum_{k=0}^{n} \left[{n \atop k}\right] (-1)^n \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}$$

([10, Theorem 4]).

Hence, it is natural to define two types of *incomplete poly-Cauchy polynomials of the second kind* $\widehat{c}_{n,\le m}^{(\mu)}(x)$ and $\widehat{c}_{n,\ge m}^{(\mu)}(x)$ by

$$\widehat{c}_{n,\le m}^{(\mu)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\le m} (-1)^n \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \tag{17}$$

$$\widehat{c}_{n,\ge m}^{(\mu)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\ge m} (-1)^n \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{(k-i+1)^\mu}, \tag{18}$$

respectively.

The generating function of $\widehat{c}_n^{(\mu)}(x)$ is given by

$$(1+x)^z \mathrm{Lif}_\mu(-\ln(1+t)) = \sum_{n=0}^{\infty} \widehat{c}_n^{(\mu)}(x) \frac{t^n}{n!} \tag{19}$$

([10, Theorem 5]. Note that z is replaced by $-z$).

The generating functions of two types of incomplete poly-Cauchy polynomials of the second kind can be also given by using the polylogarithm factorial functions $\mathrm{Lif}_\mu(z)$ in terms of $F_m(t)$. Since $F_\infty(t) = \log(1+t)$ and $F_0(t) = 0$, if we take $m \to \infty$ in (20) and $m = 1$ in (21), both of (20) and (21) are reduced to the generating function in [10, Theorem 5]. On the other hand, if $x = 0$, then Theorem 2 is reduced to [15, Theorem 2].

Theorem 2. *For integers $n \ge 1$, $m \ge 1$ and $\mu \ge 1$ we have*

$$e^{xF_m(t)} \mathrm{Lif}_\mu\bigl(-F_m(t)\bigr) = \sum_{n=0}^{\infty} \widehat{c}_{n,\le m}^{(\mu)}(x) \frac{t^n}{n!}, \tag{20}$$

$$e^{x(\log(1+t)-F_{m-1}(t))} \mathrm{Lif}_\mu\left(-\log(1+qt) + F_{m-1}(t)\right) = \sum_{n=0}^{\infty} \widehat{c}_{n,\ge m}^{(\mu)}(x) \frac{t^n}{n!}. \tag{21}$$

The generating function of two types of the incomplete Cauchy polynomials of the second kind can be also written in the form of iterated integrals.

Corollary 2. *For $\mu \ge 1$ we have*

$$\frac{e^{xF_m(t)}}{F_m(t)}\underbrace{\int_0^t \frac{1-(-t)^m}{(1+t)F_m(t)}\cdots\int_0^t \frac{1-(-t)^m}{(1+t)F_m(t)}}_{\mu-1}(1-e^{-F_m(t)})\underbrace{dt\dots dt}_{\mu-1}$$

$$=\sum_{n=0}^\infty \widehat{c}_{n,\le m}^{(\mu)}(x)\frac{t^n}{n!}, \tag{22}$$

$$\frac{e^{x(\log(1+t)-F_{m-1}(t))}}{\log(1+t)-F_{m-1}(t)}$$

$$\times\underbrace{\int_0^t \frac{(-t)^{m-1}}{(1+t)\big(\log(1+t)-F_{m-1}(t)\big)}\cdots\int_0^t \frac{(-t)^{m-1}}{(1+t)\big(\log(1+t)-F_{m-1}(t)\big)}}_{\mu-1}$$

$$\times\left(1-e^{-\log(1+t)+F_{m-1}(t)}\right)\underbrace{dt\dots dt}_{\mu-1}=\sum_{n=0}^\infty \widehat{c}_{n,\ge m}^{(\mu)}(x)\frac{t^n}{n!}. \tag{23}$$

Remark. If we take $m \to \infty$ in (22) and $m = 1$ in (23), both of (22) and (23) are reduced to the generating function in [10, Corollary 2]. If $x = 0$, then Corollary 2 are reduced to [15, Corollary 2].

5 Some properties of incomplete poly-Cauchy polynomials

It is known that poly-Bernoulli numbers satisfy the duality theorem $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \ge 0$ [11, Theorem 2] because of the symmetric formula

$$\sum_{n=0}^\infty\sum_{k=0}^\infty B_n^{(-k)}\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^{x+y}}{e^x+e^y-e^{x+y}}. \tag{24}$$

However, incomplete poly-Cauchy polynomials do not satisfy the duality theorem for any integer $m \ge 1$, by the following results.

Theorem 3. *For nonnegative integers n, k, and a positive integer m, we* *have*

$$\sum_{n=0}^\infty\sum_{k=0}^\infty c_{n,\le m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}=\exp\bigl((e^y-z)F_m(x)+y\bigr)\,, \tag{25}$$

$$\sum_{n=0}^\infty\sum_{k=0}^\infty c_{n,\ge m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}$$
$$=\exp\bigl((e^y-z)(\log(1+x)-F_{m-1}(x))+y\bigr)\,. \tag{26}$$

Remark. If $m\to\infty$ in (25) or $m=1$ in (26), both identities are reduced to the first identity in [10, Proposition 1]. If $z=0$, then Theorem 3 is reduced to [15, Theorem 4], where z is replaced by $-z$.

Similarly, incomplete poly-Cauchy polynomials of the second kind have the following properties. If $m\to\infty$ in (27) or $m=1$ in (28), both identities are reduced to the second identity in [10, Proposition 1]. If $z=0$, then Theorem 3 is reduced to [15, Theorem 5], where z is replaced by $-z$.

Theorem 4. *For nonnegative integers n, k, and a positive integer m, we have*

$$\sum_{n=0}^\infty\sum_{k=0}^\infty \widehat{c}_{n,\le m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}=\exp\bigl((z-e^y)F_m(x)+y\bigr)\,, \tag{27}$$

$$\sum_{n=0}^\infty\sum_{k=0}^\infty \widehat{c}_{n,\ge m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}$$
$$=\exp\bigl((z-e^y)\bigl(\log(1+x)-F_{m-1}(x)\bigr)+y\bigr)\,. \tag{28}$$

By using Theorem 3, we have explicit expressions of incomplete poly-Cauchy polynomials of the first kind with negative indices.

Theorem 5. *For nonnegative integers n, k, and a positive integer m, we* *have*

$$c_{n,\le m}^{(-k)}(z)=\sum_{j=0}^{k}\sum_{i=0}^{n}\sum_{l=0}^{i} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\binom{i}{l}\left[{n \atop i+j} \right]_{\le m}(-1)^{n-i-i+l}z^l, \tag{29}$$

$$c_{n,\ge m}^{(-k)}(z)=\sum_{j=0}^{k}\sum_{i=0}^{n}\sum_{l=0}^{i} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\binom{i}{l}\left[{n \atop i+j} \right]_{\ge m}(-1)^{n-i-i+l}z^l. \tag{30}$$

Remark. If $m\to\infty$ in (29), or if $m=1$ in (30), both identities in Theorem 5 are reduced to the first identity in [10, Theorem 8], where z is replaced by $-z$, and only classical Stirling numbers of both kinds are used.

If $z=0$, then we have the following. These are different expressions seen in [15, Theorem 6].

Corollary 3. *For nonnegative integers n, k, and a positive integer m, we have*

$$c_{n,\le m}^{(-k)}=\sum_{j=0}^{k}\sum_{i=0}^{n} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\left[{n \atop i+j} \right]_{\le m}(-1)^{n-i-j}, \tag{31}$$

$$c_{n,\ge m}^{(-k)}=\sum_{j=0}^{k}\sum_{i=0}^{n} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\left[{n \atop i+j} \right]_{\ge m}(-1)^{n-i-j}. \tag{32}$$

Similarly, using Theorem 4, we have explicit expressions of incomplete poly-Cauchy polynomials of the second kind with negative indices. If $m\to\infty$ in (33), or if $m=1$ in (34), any identity in Theorem 6 is reduced to the second identity in [10, Theorem 8], where z is replaced by $-z$, and only classical Stirling numbers of both kinds are used.

Theorem 6. *For nonnegative integers n, k, and a positive integer m, we have*

$$\widehat{c}_{n,\le m}^{(-k)}(z)=\sum_{j=0}^{k}\sum_{i=0}^{n}\sum_{l=0}^{i} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\binom{i}{l}\left[{n \atop i+j} \right]_{\le m}(-1)^{n-l}z^l, \tag{33}$$

$$\widehat{c}_{n,\ge m}^{(-k)}(z)=\sum_{j=0}^{k}\sum_{i=0}^{n}\sum_{l=0}^{i} j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\binom{i}{l}\left[{n \atop i+j} \right]_{\ge m}(-1)^{n-l}z^l. \tag{34}$$

If $z=0$, then then we have the following. These are different expressions seen in [15, Theorem 7].

Corollary 4. *For nonnegative integers n, k, and a positive integer m, we have*

$$\widehat{c}_{n,\le m}^{(-k)}=(-1)^n\sum_{j=0}^k\sum_{i=0}^n j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\left[{n \atop i+j} \right]_{\le m}, \tag{35}$$

$$\widehat{c}_{n,\ge m}^{(-k)}=(-1)^n\sum_{j=0}^k\sum_{i=0}^n j!\left\{ {k+1 \atop j+1} \right\}\binom{i+j}{j}\left[{n \atop i+j} \right]_{\ge m}. \tag{36}$$

6 Incomplete poly-Bernoulli polynomials

In [6], poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ are defined in (1). In [1, 2], x and $-x$ are interchanged in the definition of $B_n^{(\mu)}(x)$. In [3], an extended concept named poly-Bernoulli polynomials with a q parameter is introduced. In [16], still different poly-Bernoulli polynomials $B_n^{(\mu)}(x)$ are defined. Poly-Bernoulli polynomials have an explicit expression in terms of the Stirling numbers of the second kind:

$$B_n^{(\mu)}(x)=\sum_{k=0}^n\frac{(-1)^{n-k}k!}{(k+1)^\mu}\sum_{l=0}^n\binom{n}{l}\left\{ {l \atop k} \right\}x^{n-l}. \tag{37}$$

When $x=0$, we get the expression of poly-Bernoulli numbers $B_n^{(\mu)}(0)=B_n^{(\mu)}$:

$$B_n^{(\mu)}=\sum_{k=0}^n\frac{(-1)^{n-k}k!}{(k+1)^\mu}\left\{ {n \atop k} \right\}$$

([11, Theorem 1]).

By using two types of incomplete Stirling numbers, define *restricted poly-Bernoulli polynomials* $B_{n,\le m}^{(\mu)}(x)$ and *associated poly-Bernoulli polynomials* $B_{n,\ge m}^{(\mu)}(x)$ by

$$B_{n,\le m}^{(\mu)}(x)=\sum_{k=0}^n\frac{(-1)^{n-k}k!}{(k+1)^\mu}\sum_{l=0}^n\binom{n}{l}\left\{ {l \atop k} \right\}_{\le m}x^{n-l}\quad (n\ge 0) \tag{38}$$

and

$$B_{n,\geq m}^{(\mu)}(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}k!}{(k+1)^{\mu}}\sum_{l=0}^{n}\binom{n}{l}\left\{ {l \atop k} \right\}_{\geq m} x^{n-l} \quad (n\geq 0)\,, \tag{39}$$

respectively. We call these numbers as *incomplete poly-Bernoulli polynomials.* If $x=0$, then they are reduced to restricted poly-Bernoulli numbers and associated poly-Bernoulli numbers, respectively ([18]).

One can deduce that the numbers $B_{n,\leq m}^{(\mu)}(x)$ and $B_{n,\geq m}^{(\mu)}(x)$ have the generating functions by using the polylogarithms.

Theorem 7. *We have*

$$\frac{\mathrm{Li}_{\mu}\big(1-E_m(-t)\big)}{1-E_m(-t)}e^{-tx}=\sum_{n=0}^{\infty}B_{n,\leq m}^{(\mu)}(x)\frac{t^n}{n!} \tag{40}$$

and

$$\frac{\mathrm{Li}_{\mu}\big(E_{m-1}(-t)-e^{-t}\big)}{E_{m-1}(-t)-e^{-t}}e^{-tx}=\sum_{n=0}^{\infty}B_{n,\geq m}^{(\mu)}(x)\frac{t^n}{n!}\,. \tag{41}$$

Remark. If $x=0$, they are reduced to the generating functions in [18, Theorem 1].

If $m\to\infty$ in (40) or $m=1$ in (41), they are reduced to the generating function in [3, Theorem 2.1] by putting $q=1$.

For $\mu\geq 1$, the generating functions can be written in the form of iterated integrals. We set $E_{-1}(-t)=0$ for convenience.

Theorem 8.

$$\frac{e^{-tx}}{1-E_m(-t)}\cdot\underbrace{\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}\cdots\int_0^t\frac{E_{m-1}(-t)}{1-E_m(-t)}}_{\mu-1}\left(-\log\left(E_m(-t)\right)\right)\underbrace{dt\dots dt}_{\mu-1}$$

$$=\sum_{n=0}^{\infty}B_{n,\le m}^{(\mu)}(x)\frac{t^n}{n!}, \tag{42}$$

$$\frac{e^{-tx}}{E_{m-1}(-t)-e^{-t}}\cdot\underbrace{\int_0^t\frac{e^{-t}-E_{m-2}(-t)}{E_{m-1}(-t)-e^{-t}}\cdots\int_0^t\frac{e^{-t}-E_{m-2}(-t)}{E_{m-1}(-t)-e^{-t}}}_{\mu-1}$$

$$\times\left(-\log\left(1+e^{-t}-E_{m-1}(-t)\right)\right)\underbrace{dt\dots dt}_{\mu-1}$$

$$=\sum_{n=0}^{\infty}B_{n,\ge m}^{(\mu)}(x)\frac{t^n}{n!}. \tag{43}$$

Remark. If $x=0$, then Theorem 8 is reduced to [18, Theorem 2]. In addition, if $m\to\infty$ in (42), by $E_\infty(-t)=e^{-t}$, and if $m=1$ in (43), both of them are reduced to the iterated form (2) in ([11]) by setting $t=0$.

A symmetric property does not hold for generalized poly-Bernoulli numbers with negative indices because of the following.

Theorem 9. *We have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B_{n,\le m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^{y-xz}}{1-e^y\left(1-E_m(-x)\right)}, \tag{44}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B_{n,\ge m}^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!}=\frac{e^{y-xz}}{1-e^y\left(E_{m-1}(-x)-e^{-x}\right)}. \tag{45}$$

Remark. If $m\to\infty$ in (44) or $m=1$ in (45), both identities are reduced to (24) with $z=0$.

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